

The generalized Chapman-Richards function and applications to tree and stand growth

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Abstract: The generalized Chapman-Richards model was derived from the Chapman-Richards function in which parameters η , κ and m were unconstrained. Based on the structure of solutions and biological interpretations, the model could be classified into eight cases (three categories) at all and among them only 4 kinds of cases are suitable in forestry that represent four typical growth patterns of trees and stands. For each of 4 equations, the model properties and biological interpretations for parameters were discussed in detail. The generalized Chapman-Richards model was capable of describing a wide range of growth curves that was asymptotic or nonasymptotic, with or without inflection point. In order to illustrate the versatility of the model, it was fitted to a group of data sets concerning the DBH growth of cryptomeria plantations with 4 initial densities and the DBH and height growth of natural Korean pine tree. Comparing the generalized Chapman-Richards function and the Schnute model, it was found that the parameters and expressions of the two models were interchangeable in theory, and the fitting results were explicitly identical in empirical applications.

Keywords: Generalized Chapman-Richards function; Schnute model; Growth model; Growth pattern; *Cryptomeria japonica*; *Pinus koraiensis*

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Introduction

The Chapman-Richards function, based on the Von Bertalanffy's growth theory (Bertalanffy, 1957), embodies such commonly used growth functions as monomolecular, Gompertz, and logistic equations. The model has been widely applied in forestry thanks to its flexibility, accuracy, and meaningful analytical properties (Cooper 1961; Pienaar and Turnbull 1973; Osumi and Ishikawa 1983; Ito and Osumi 1984). Nevertheless, the model's properties and features were not discussed in detail when it was first introduced to forestry. For example, the estimated value of parameter m of the model was sometimes negative ($m < 0$). In such a case, the original assumptions for the model fail to be valid any more. When the model was used as height growth curve, the estimated values of m were less than 0 and the inflection points were not existent (Zhang and Wang 1992; Li 1995). A question is naturally raised that whether the model in such special cases reflects tree growth in reality.

Based on the biological theory of accelerating growth, Schnute (1981) built a comprehensive growth model with statistically stable parameters that was claimed to cover all other growth functions existing to date. The Schnute model

has been accepted in forestry quickly because of its versatility and stable parameter estimates (Bredenkamp and Gregoire, 1988; Yang and Feng, 1989; Zeide, 1993; Feng, 1997). Bredenkamp and Gregoire (1988) and Yang and Feng (1989) studied the performance of the Chapman-Richards function along with the Schnute model. They concluded that stands resumed growth after the competition mortality beyond what appeared to be an asymptotic level where the Chapman-Richards function cannot fit the data satisfactorily. In contrast, the Schnute model can successfully tracked such a renewed growth of stands after marked competition mortality. Feng (1997) applied the Schnute model to investigating stand growth of different initial spacing, trying to find biological meanings for the parameters. The reason to draw the conclusion is probably because the model was only used for data fitting rather than investigating the biological and local asymptotic properties of parameters.

In this study, the generalized Chapman-Richards function was derived and analyzed theoretically as an extended form of the Chapman-Richards growth function and particular attention is given to the four kinds of growth patterns of the function in empirical applications to tree and stand growth, facilitating comparisons between growth curves of different shapes. Comparisons of this growth function with the Schnute model were also discussed in detail. It was demonstrated that the generalized Chapman-Richards function has the same capabilities as the Schnute model does and the two models can fit the same data set equally well with the identical accuracy for describing the quadratic mean diameter growth of cryptomeria (*Cryptomeria japonica* D. Don.) stands with different initial spacing and the

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DBH and height growth for a dominant tree of natural Korean pine (*Pinus koraiensis* Sib. et Zucc.).

The generalized Chapman-Richards function

The Chapman-Richards equation was derived from the Bertalanffy equation "When limitations imposed by its theoretical background are discarded" (Richards 1959). Parameter m in the equation was assumed to be positive, not only in range of $0 < m < 1$ as in Bertalanffy's case. The theoretical growth equations of Mitscherlich, Gompertz and the logistic are special cases of the function for parameter $m = 0$, $m \rightarrow 1$, and $m = 2$, respectively. By studying plant growth, Richards (1959) proposed to extend the range of parameter m in Von Bertalanffy's growth model to $m > 0$ rather than $0 < m < 1$:

$$\frac{dy}{dt} = \eta y^m - \kappa y \quad (1)$$

where y is a measure of plant size; t is plant age; η , κ and m are parameters.

It is assumed that $\eta > 0$ and $\kappa > 0$ for $0 < m < 1$, where η and κ are constants of anabolism and catabolism, respectively, and $\eta < 0$ and $\kappa < 0$ for $m > 1$, where η and κ are constants of catabolism and anabolism, respectively.

In addition to the range of $m > 0$, the two exceptions could occur in applying the Chapman-Richards function to tree and stand growth. One occurs when $m < 0$ but $\eta > 0$ and $\kappa > 0$; the other occurs when $m < 0$ but $\eta > 0$ and $\kappa < 0$. To deal with the two exceptional cases, we need to investigate more generalized form of the Chapman-Richards function.

Integral form

In equation (1), parameters η , κ and m were assumed to take any value without being restricted to $m > 0$. However, since growth rate of a tree or a stand, dy/dt , has to be greater than or equal to zero, $\eta < 0$ and $\kappa > 0$ can't occur for any value of m . If parameters η , κ , and m in equation (1) are unconstrained, it will be

$$\frac{dy}{dt} = \eta y^m - \kappa y \quad \eta, \kappa, \text{ and } m \text{ are any constants} \quad (2)$$

In equation (2), $\eta < 0$ and $\kappa > 0$ cannot be true simultaneously for any m .

The integral form of equation (2) with initial condition $y(t_0) = y_0$ is

$$y(t) = \left[\frac{\eta}{\kappa} - \left(\frac{\eta}{\kappa} - y_0^{1-m} \right) e^{-\kappa(1-m)(t-t_0)} \right]^{\frac{1}{1-m}} \quad (3)$$

where t_0 and y_0 are initial age and size, respectively. Equation (3) can be rewritten as

$$y(t) = A \left[\pm \left(1 - B e^{-\kappa t} \right) \right]^{\frac{1}{1-m}} \quad \begin{cases} + & \text{if } \eta/\kappa > 0 \\ - & \text{otherwise} \end{cases} \quad (4)$$

where

$$A = \left| \frac{\eta}{\kappa} \right|^{\frac{1}{1-m}}, B = \left[1 - \left(\frac{y_0}{\eta/\kappa} \right)^{1-m} \right] e^{\kappa t_0}, K = \kappa(1-m)$$

Therefore, the integral form of the generalized Chapman-Richards function has two basic forms. One is the first case of equation (4) that we have known very well and another is the second case of equation (4) that was discussed by Li *et al.* (1993).

Relationships between Parameters

The equation (3) may or may not define a curve that crosses the t -axis, has an inflection point, or exhibits asymptotic behavior. The equation does possess the pre-mentioned three parameters, then the parameters approximating to size 0, asymptotic size, and inflection point are defined as follows.

1) Cross-over point: If there exist a time, τ_0 , such that $y(\tau_0) = 0$, then by solving for τ_0 from equation (3), we have

$$\tau_0 = t_0 - \frac{1}{\kappa(1-m)} \log \left[\frac{\frac{\eta}{\kappa}}{\frac{\eta}{\kappa} - y_0^{1-m}} \right] \quad (5)$$

2) Asymptotic value: the asymptotic size as $t \rightarrow \infty$ obtains from equation (3) as

$$y_\infty = \left[\frac{\eta}{\kappa} \right]^{\frac{1}{1-m}} \quad (6)$$

3) Inflection point: This point connects regions of accelerating and decelerating growth change. Location of inflection point (t_i) and corresponding size $y(t_i)$ (defined as y_i) can

be found from $\frac{d^2 y}{dt^2} = 0$. Conditions of the existence of

inflection points follow are $t_i > t_0$ and $y_i > y_0$. The second order derivative of equation (3) is

$$\frac{d^2 y}{dt^2} = \kappa \frac{dy}{dt} \left(m \frac{\eta}{\kappa} y^{m-1} - 1 \right) \quad (7)$$

If there is a time, t_i , such that equation (6) equals zero, then inflection point is at

$$t_l = t_0 - \frac{1}{\kappa(1-m)} \log \left[\frac{(1-m)\frac{\eta}{\kappa}}{\frac{\eta}{\kappa} - y_0^{1-m}} \right] \quad (8)$$

$$y_l = \left[m \frac{\eta}{\kappa} \right]^{\frac{1}{1-m}} \quad (9)$$

If a particular curve has these properties, then the curve can be fully represented by equations (5), (6), (8), and (9). Whether these properties exist or not depends on what kind

of value the parameters in equation (3) take.

Model Properties

Parameters τ_0 , y_∞ , t_l , and y_l defined by equations (5), (6), (8), and (9) are meaningful only if the expressions in the square brackets are positive. The integral form of the generalized Chapman-Richards function has eight possible cases for different growth curves that are primarily determined by the three parameters η , κ , and m in equation (2). All possible solutions and its characteristics are summarized in Table 1.

Table 1. Basic characteristics for the solutions of the generalized Chapman-Richards function

Cases	κ	η	m	Equation	Cross-over point (τ_0)	Asymptotic value (y_∞)	Inflection point (t_l, y_l)	Existing equations	Cases by Schnute (1981)
Case 1	$\kappa > 0$	$\eta > 0$	$0 < m < 1$	(10) (14)	Existence None	Existence Existence	Existence Existence	Chapman-Richards Gompertz ($m \rightarrow 1$)	case 1
Case 2	$\kappa > 0$	$\eta > 0$	$m \leq 0$	(11)	Existence	Existence	None	monomolecular ($m=0$)	case 2
Case 3	$\kappa > 0$	$\eta > 0$	$m > 1$	(15)	None	None	None	None	case 6
Case 4	$\kappa < 0$	$\eta > 0$	$0 \leq m < 1$	(16)	Existence	None	None	None	case 4
Case 5	$\kappa < 0$	$\eta > 0$	$m < 0$	(12)	Existence	None	Existence	None	case 3
Case 6	$\kappa < 0$	$\eta > 0$	$m > 1$	(17)	None	None	None	None	case 7
Case 7	$\kappa < 0$	$\eta < 0$	$m < 1$	(18)	None	None	None	None	case 5
Case 8	$\kappa < 0$	$\eta < 0$	$m > 1$	(13)	Existence	Existence	Existence	logistic ($m=2$)	case 8

All eight cases listed in Table 1 belong to the three categories associated with the combination of the parameter value and each category has its own expression to be discussed later. Category 1 for $\kappa > 0$ and $\eta > 0$ covers cases 1, 2 and 3. Category 2 includes cases 4, 5 and 6 with $\kappa > 0$ and $\eta < 0$. Cases 7 and 8 are of category 3 with $\kappa < 0$ and $\eta < 0$. However, it is noted some cases are not suitable to describe growth dynamics because almost all growth models used in forestry management are data-based (Shvets and Zeide, 1996). In application, only following four cases of the generalized Chapman-Richards function are useful to coincide with our ecological and biological knowledge.

Case 1: $\kappa > 0$, $\eta > 0$ and $0 < m < 1$ (category 1)

This kind of model represents the classical situation of Chapman-Richards function in which all parameters have biological meanings (Richards, 1959) and parameters τ_0 , y_∞ , t_l , and y_l are defined (Table 2). The curve is sigmoid and asymptotic with limiting size, y_∞ . The corresponding growth function is

$$y(t) = A(1 - Be^{-Kt})^{\frac{1}{1-m}} \quad A, B, K > 0 \text{ and } 0 < m < 1 \quad (10)$$

In equation (10), A is the final (asymptotic) size of tree or stand yield ($A = y_\infty$). B is related to initial size (y_0) and can be thought as the "intercept" on y -axis for $t_0 = 0$, $B = 1$ when $t_0 = 0$ and $y_0 = 0$. K reflects growth rate of trees when $y(t)$

increases from y_0 to final size, y_∞ . Parameter m is the power exponent of tree size for the anabolism rate, which determines curve shapes and the location of an inflection point.

Case 2: $\kappa > 0$, $\eta > 0$ and $m < 0$ (category 1)

For this case, growth rate (dy/dt) has two components: anabolism part (ηy^m) and catabolism part (κy). Parameters η and κ are for anabolism and catabolism rate, respectively. Parameter $m \leq 0$ means that the photosynthetic activity decreases as size, $y(t)$, increases, but constructive metabolism is larger than destructive one. As a result, the growth rate maintains the maximum value at initial stage (t_0), then decreases until the final state and $dy/dt \geq 0$. The yield curve has no period of accelerating growth. However, the decelerating growth takes place in the beginning and lasts until the final limited size, y_∞ (see Fig. 1). The growth function can be expressed as

$$y(t) = A(1 - Be^{-Kt})^{\frac{1}{1-m}} \quad A, B, K > 0 \text{ and } m \leq 0 \quad (11)$$

In this equation, all parameters have the same biological meanings as those in equation (10).

In this case, the growth curve is asymptotic and crosses the t -axis, but it has no inflection point because it does not meet the condition as in case 1 (see Table 2). Even though equation (11) seems presenting a rather unrealistic pattern of growth, some mean diameter or height data of

fast-growing species can be modeled more or less by this equation because those species keep growing from an earlier age with no inflection point as shown in Figure 1. When $m = 0$, the equation becomes the monomolecular or the Mitscherlich equation (Richards, 1959).

Case 5: $\kappa < 0$, $\eta > 0$ and $m \leq 0$ (category 2)

When equation (2) has the parameters of this case, it is hard to give the clear ecological interpretation for tree growth. But, it may be reasonable for us to consider it as an empirical growth model because the growth rate of tree or stand is greater than 0, $dy/dt > 0$, after all.

In this case, the curve is neither asymptotic nor unbounded. This kind of growth curve crosses the t -axis at age τ_0 and has an inflection point (Table 2). The growth decelerates in the period of $\tau_0 \sim t_i$ and accelerates forever afterwards. The shape of growth rate, dy/dt , is "V-form", which is a reversed shape of general growth curve (e.g. case 1), and it reaches its minimum value at inflection point. Such a growth curve is analogous to the one shown in Figure 2. The total growth curve in period of $\tau_0 \sim t_i$ is similar to the one in case 2 and then blows-up again. The corresponding equation becomes

$$y(t) = A \left[- \left(1 - Be^{-Kt} \right) \right]^{\frac{1}{1-m}} \quad A, B > 0, K < 0, \text{ and } m \leq 0 \quad (12)$$

In equation (12), the parameter A has no exact biological meaning and no longer represents asymptotic size, because $A \rightarrow \infty$ as $t \rightarrow \infty$. B is related to y_0 . K determines growth rate.

Such a special growth curve may not occur so often, but it is easy to understand circumstances. If a stand of fast-growing species with high initial density experiences a strong intra-competition, then the stand will re-start growth after competition mortality period is over. Such a kind of growth should be well described by equation (12) for the pattern of a stand resuming growth beyond what had appeared to be an asymptotic level after marked competition mortality (Li et al., 1988).

Case 8: $\kappa < 0$, $\eta < 0$ and $m > 1$ (category 3)

Growth rate (dy/dt) is composed of two parts: anabolism part (κy) and catabolism part (ηy^m). Parameters κ and η stand for anabolism and catabolism rate, respectively. Parameter m is the power exponent of size for catabolism rate. The function is

$$y(t) = A(1 - Be^{-Kt})^{\frac{1}{1-m}} \quad A, K > 0, B < 0, \text{ and } m > 1 \quad (13)$$

where A , B and K have same meanings as those in equation (10).

Table 2. Properties for some solutions of the generalized Chapman-Richards function in tree and stand growth

Cases	Equation	Cross-over point (τ_0)	Asymptotic value (y_∞)	Inflection point		
				t_i	y_i	Condition of existence
Case 1	(10)	$\frac{\ln(B)}{K}$	A	$\frac{1}{K} \ln \left(\frac{B}{1-m} \right)$	$Am^{\frac{1}{1-m}}$	$m > \left(\frac{y_0}{A} \right)^{1-m}$
Case 2	(11)	$\frac{\ln(B)}{K}$	A	None	None	
Case 5	(12)	$-\frac{\ln(B)}{ K }$	None	$-\frac{1}{ K } \ln \left(\frac{B}{1-m} \right)$	$A m ^{\frac{1}{1-m}}$	$ m > \left(\frac{y_0}{A} \right)^{1-m}$
Case 8	(13)	None	A	$\frac{1}{K} \ln \left(\frac{ B }{m-1} \right)$	$Am^{\frac{1}{m-1}}$	$m < \left(\frac{y_0}{A} \right)^{-(m-1)}$
Gompertz	(14)	None	A	$\frac{1}{K} \ln(B)$	A/e	$A > ey_0$

Note: $y_0 = y(t_0)$, for example, $y_0 = A(1 - Be^{-Kt_0})^{\frac{1}{1-m}}$ for equation (10).

The function forms a sigmoid and asymptotic curve with an inflection point beyond A/e ($y_i > A/e$) as Figure 3 shows. However, the curve can't be extrapolated back to the t -axis. Instead, it has the t -axis as a lower asymptote. It was reported that such a curve was not suitable in modelling tree or stand growth (Osumi and Ishikawa, 1983; Li, 1995) and known to be preferable to describe animal growth, organ growth of a tree (for example branch, leaf, and etc.), and plant growth or population dynamics with initial value (Richards, 1959; Osumi and Ishikawa, 1983; Ratkowsky

1983). However, this kind of growth curve does exist in tree growth for slow-growing species of natural forest, e.g. for natural Korean pine, as shown in next content.

The Gompertz equation ($\kappa > 0$, $\eta > 0$ and $m \rightarrow 1$) is a special example of case 1 with $m \rightarrow 1$. Properties of the Gompertz function are listed in Table 2. Many researchers found that the Gompertz equation was appropriate to use in biology as well as in forestry (Zeide, 1993). The Gompertz equation is

$$y(t) = A \exp(-Be^{-Kt}) \quad A, B, \text{ and } K > 0 \quad (14)$$

where, A is the asymptote; m determines the shape and the location of inflection point; K is a parameter to determine growth rate.

There are other four cases of the generalized Chapman-Richards function, which is not discussed in this paper because they are irrelevant to tree or stand growth. The responsible equations are

Case 3:

$$y(t) = A(1 - Be^{-Kt})^{\frac{1}{1-m}} \quad (15)$$

$A, B > 0, K < 0 \text{ and } m > 1$

Case 4:

$$y(t) = A \left[1 - (1 - Be^{-Kt})^{\frac{1}{1-m}} \right] \quad (16)$$

$A, B > 0, K < 0 \text{ and } 0 \leq m < 1$

Case 6

$$y(t) = A \left[1 - (1 - Be^{-Kt})^{\frac{1}{1-m}} \right]^{\frac{1}{1-m}} \quad (17)$$

$A, B, K > 0 \text{ and } m > 1$

Case 7:

$$y(t) = A(1 - Be^{-Kt})^{\frac{1}{1-m}} \quad (18)$$

$A > 0, B < 0, K < 0, \text{ and } m < 1$

Compare with the Schnute Growth Model

The Schnute model was developed from growth acceleration (Schnute, 1981). The model, capable of describing asymptotic as well as nonasymptotic growth trend, can be written as:

$$y(t) = (y_1^b + (y_2^b - y_1^b) \frac{1 - e^{-a(t-T_1)}}{1 - e^{-a(T_2-T_1)}})^{\frac{1}{b}} \quad (19)$$

where T_1 and T_2 are initial and terminal ages; y_1 and y_2 are

corresponding sizes at T_1 and T_2 .

Depending primarily on parameters a and b , the Schnute model was divided into eight possible characteristics shapes for growth curves and its properties were discussed case by case by Schute (1981).

By comparing assumptions, solutions and relationships among the parameters for two models, we could find that the generalized Chapman-Richards function and the Schnute model are identical.

Identical differential forms

The assumption of the generalized Chapman-Richards function can be derived from the assumption for the Schnute model and vice-versa (Li et. al., 1997).

Identical solution sets and expressions

It is shown in Li et. al.(1997) that integral solutions of two differential equations without any initial / boundary values have same formula and their parameters are interchangeable.

This analysis reveals that if the generalized Chapman-Richards model is integrated with two specific values for a growth process, then the Schnute model can be deduced from equation (2).

The relationships between two sets of parameters

Four parameters a, b, y_1 , and y_2 in the Schnute model (19) can be represented by those parameters A, B, K , and m of the generalized Chapman-Richards function (4), and vice-versa, as shown in Table 3.

The relation between two model's parameters τ_0, y_∞, t_i , and y_i are obvious under the relationships of parameters between the Schnute model and the generalized Chapman-Richards function.

In addition, each case of the generalized Chapman-Richards function matches one of eight cases of the Schnute model (Schnute, 1981) as shown in Table 1. All of eight equations (10), (11), (12), (16), (18), (15), (17), (13) and its properties correspond to the curves 1 to 8 in Fig. 2, established by Schnute (1981).

Table 3. Relationships of the parameters between the Schnute model and the generalized Chapman-Richards function

Parameters of the Schnute model	Parameters of the generalized Chapman-Richards function
$a = K = \kappa(1 - m)$	$K = a$
$b = 1 - m$	$m = 1 - b$
$y_1 = A \left[\pm (1 - Be^{-Kt_1}) \right]^{\frac{1}{1-m}} \quad \begin{cases} - & \text{for cases 4, 5, 6} \\ + & \text{otherwise} \end{cases}$	$A = \left[\pm \left(y_1^b + \frac{y_2^b - y_1^b}{1 - e^{-a(T_2 - T_1)}} \right) \right]^{\frac{1}{b}} \quad \begin{cases} - & \text{for cases 4, 5, 6} \\ + & \text{otherwise} \end{cases}$
$y_2 = A \left[\pm (1 - Be^{-Kt_2}) \right]^{\frac{1}{1-m}} \quad \begin{cases} - & \text{for cases 4, 5, 6} \\ + & \text{otherwise} \end{cases}$	$B = \frac{e^{a(T_2 + T_1)} (y_2^b - y_1^b)}{e^{aT_2} y_2^b - e^{aT_1} y_1^b}$

A practical example

The growth data published by Yang and Feng (1989) and Feng (1997) provide reliable descriptions of the quadratic mean diameter (*DBH*) growth for cryptomeria plantations with different spacing. The base for this study are four kinds of initial spacing 3.1 m×3.1 m, 2.3 m×2.3 m, 1.7 m×1.7 m, 1.5 m×1.5 m (1 024, 1 934, 3 520, 4 552 stems per hectare, respectively) in the permanent plots of 0.1 hm² in sizes. The *DBH* was first measured at age of 6 and there had been 19 measurements from 1930 to 1956. The last measurements were made at age of 26.

Korean pine is a dominant species in the temperate hardwood-conifer forest, northeastern China. It is slow growing tree and 250-year-old trees in virgin forests commonly vary from 28 to 30 m in height, and from 45 to 50 cm in diameter (Li, 1997). Rated as intermediate in tolerance, it is known as shade-tolerant at young stage, but gradually turns to be intolerant in later life. The *DBH* and height growth data for a dominant tree of natural Korean pine from Xiaoxing'an Mountain, northeast China are listed in Table 4. The data were used to fit case 8 (Equation (13)) of the generalized Chapman-Richards function.

We performed a nonlinear regression analysis using Marquardt's method implemented in SAS 6.12 software (SAS Institute Inc., 1990). Because the generalized Chapman-Richards model (4) has two kinds of forms, the two expressions were fitted to same data set simultane-

ously. One, which converged to stable parameter estimates and resulted in the minimum residual sum of squares (*RSS*) or mean squared error (*MSE*), was the final equation to select. The estimated parameters and fit statistics of the generalized Chapman-Richards function and the Schnute model for the *DBH* growth of the cryptomeria plantations and the *DBH* and height growth of the natural Korean pine are summarized in Table 5.

Table 4. *DBH* and height growth data for a 265-year-old tree of natural Korean pine in northeast China

Age (yrs)	<i>DBH</i> (cm)	Height (m)	Age (yrs)	<i>DBH</i> (cm)	Height (m)
10	0.0	0.42	150	15.5	17.35
20	0.0	1.06	160	17.9	19.05
30	1.1	1.70	170	22.0	20.95
40	2.0	2.54	180	27.0	22.90
50	2.5	3.20	190	30.8	25.00
60	3.4	3.93	200	34.1	26.55
70	3.9	4.58	210	37.1	27.95
80	4.6	5.20	220	40.7	29.12
90	5.5	6.15	230	44.0	30.15
100	6.7	7.48	240	40.7	31.05
110	7.6	8.85	250	44.0	31.95
120	9.4	10.35	260	55.4	32.65
130	11.2	12.25	265	57.9	32.90
140	13.1	15.20			

Table 5. Parameter estimates for the generalized Chapman-Richards function and Schnute model to describe *DBH* growth of Cryptomeria stands planted at four kinds of spacing, and *DBH* and height growth for a dominant tree of the natural Korean pine

The generalized Chapman-Richards function											
Spacing	Equation	<i>A</i>	<i>B</i>	<i>K</i>	<i>m</i>	<i>RSS</i>	<i>MSE</i>	τ_0	y_∞	t_i	y_i
3.1 × 3.1	(10)	28.74364	1.64909	0.13537	0.097665	1.1752	0.0783	3.6952	28.7436	4.45	2.1824
2.3 × 2.3	(11)	24.30352	1.68105	0.10742	-0.48008	0.9399	0.0627	4.8353	24.3035	None	None
1.7 × 1.7	(11)	23.98952	1.35453	0.063638	-0.57707	1.2435	0.0829	4.7685	23.9895	None	None
1.5 × 1.5	(12)	14.08791	0.65694	-0.071539	-2.42242	2.7610	0.1841	5.8731	None	23.07	18.2440
<i>DBH</i>	(13)	81.91948	-25.48801	0.017037	1.67657	13.4964	0.5868	None	81.91948	213.00	38.1665
Height	(13)	34.80487	-51.32716	0.025043	2.08460	2.5428	0.1106	None	34.8049	154.02	17.6808
The Schnute model ^a											
Spacing	Case by Schnute	y_1	y_2	<i>a</i>	<i>b</i>	<i>RSS</i>	<i>MSE</i>	τ_0	y_∞	t_i	y_i
3.1 × 3.1	1	6.68083	27.19243	0.13537	0.902335	1.1752	0.0783	3.6952	28.7436	4.45	2.1824
2.3 × 2.3	2	5.72297	22.58357	0.10742	1.48008	0.9399	0.0627	4.8353	24.3035	None	None
1.7 × 1.7	2	4.65698	19.83783	0.063638	1.57707	1.2435	0.0829	4.7685	23.9895	None	None
1.5 × 1.5	3	3.57072	19.82632	-0.071539	3.42242	2.7610	0.1841	5.8731	None	23.07	18.2440
<i>DBH</i>	8	1.67597	56.94357	0.017037	0.67656	13.4964	0.5868	None	81.91948	213.00	38.1665
Height	8	1.13523	32.77536	0.025043	-1.08460	2.5428	0.1106	None	34.8049	154.02	17.6808

^a $T_1=6$ and $T_2=26$ for *DBH* data of cryptomeria stands, while $T_1=10$ and $T_2=265$ for height data of the Korean pine tree.

The fitted curves to six data sets represent all of the four characteristic growth patterns (three categories) found in the generalized Chapman-Richards function for modeling tree and stand growth or biological growth. There are three kinds of growth curves (case 1, 2, and 5) to describe the

DBH growth of the cryptomeria stands with different initial densities, and one (case 8) to the *DBH* and height growth of the Korean pine tree.

The *DBH* growth curve of cryptomeria stand with spacing of 3.1m×3.1m belongs to case 1 of the generalized

Chapman-Richards model or equation (10) and there are inflection point, asymptotic size, and τ_0 . For stands with spacing of 2.3 m \times 2.3 m and 1.7 m \times 1.7 m, only asymptote and τ_0 exist without inflection point because of $m < 0$, which exactly belongs to case 2 of the generalized Chapman-Richards or equation (11) (see Fig. 1).

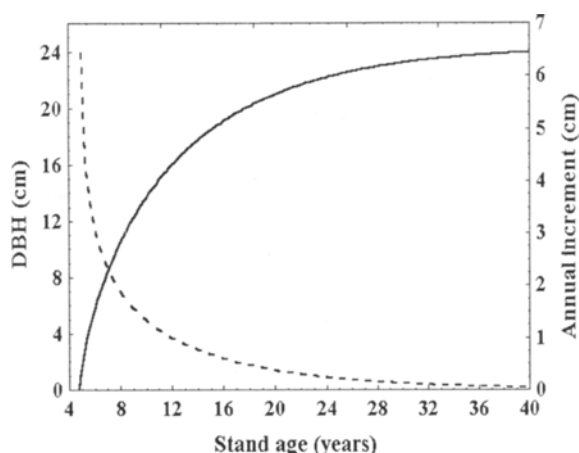


Fig. 1 The DBH growth curve of *Cryptomeria japonica* with the initial spacing of 2.3 m \times 2.3 m (1934 stems/ha) in Table 4 predicted by equation (11).

The solid line indicates the cumulative growth curve and the dashed line indicates the current annual growth curve.

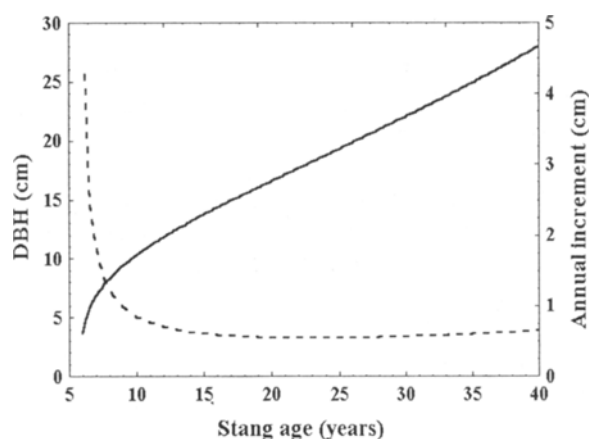


Fig. 2 The DBH growth curve of *Cryptomeria japonica* with the initial spacing of 1.5 m \times 1.5 m (4552 stems/ha) in Table 4 predicted by equation (12)

The solid line indicates the cumulative growth curve and the dashed line indicates the current annual growth curve.

Bredenkamp and Gregoire (1988) reported that the four-parameters Chapman-Richards function was not appropriate any more for fitting DBH data of *Eucalyptus grandis* plantations for the last three densities (2965-6726 stems/hm²) because the stand resumed growth after competition mortality. Actually, the kinds of growth curves belong to case 5 of the generalized Chapman-Richards model

or equation (12), which was reported early by Li *et al.* (1993). Equation (10) does not describe such data very well and the right choice for these stands should be equation (12). Again, such situation happens in DBH growth curve of cryptomeria plantation with initial spacing of 1.5 m \times 1.5 m (4552 trees/hm²) as shown in Fig. 2. The growth curve crosses the t -axis at about 6 years and resumed growth after competition mortality at the age of 23 years, but no asymptote. Therefore, the case 5 of the generalized Chapman-Richards function is also capable of tracking the growth after mortality induced by competition as the Schnute model does.

Because of $m > 1$ and $K < 0$, the DBH and height growth curves of the Korean pine tree belong to case 8 of the generalized Chapman-Richards model or equation (13). The two curves, specially for height growth, are close to the logistic function in form, parameter m varying between 1.7 and 2.1, and inflection points appear at tree size grown about $A/2$ (see Table 4). The asymptotic sizes and ages of annual growth reaching maximum for fitted DBH and height growth curves are explicitly coincided with real growth, which reported by Li (1997) that the maximum annual growth occurs from 120 to 140 years in height, and from 200 to 220 years in DBH for natural Korean pine trees. The maximal growth rates of tree sizes are delayed in development for natural Korean pine than other fast growing species, and such fitting curves having $m > 1$ must be legitimate. The fitted height growth curve of the Korean pine tree is shown in Fig. 3.

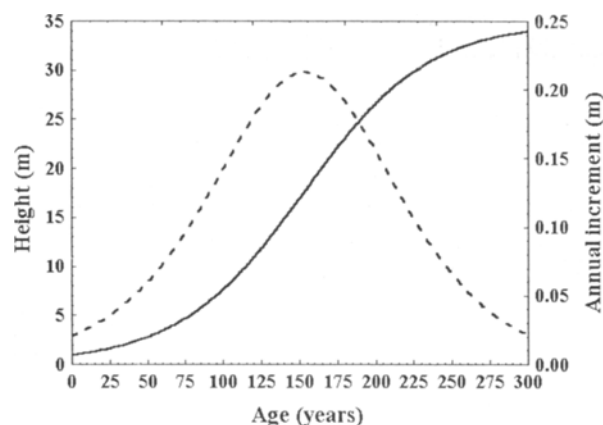


Fig. 3 The height growth curve of the natural Korean pine tree predicted by equation (13)

The solid line indicates the cumulative growth curve and the dashed line indicates the current annual growth curve.

From Table 5, it is also apparent that RSS , MSE and parameters τ_0 , y_∞ , t_i , y_1 (if they are defined) of the generalized Chapman-Richards model coincide with those of the Schnute model. Also, parameter K in the generalized Chapman-Richards model has the same value as parameter a in the Schnute model. Depending on the relationships of parameters between two models, parameters A ,

B , K , and m in the generalized Chapman-Richards function and parameters y_1 , y_2 , a , and b in the Schnute model can be precisely interchangeable. Thus, it is concluded that the two models are equivalent in theory and experience.

Conclusion

There are two basic forms for the integral form of the generalized Chapman-Richards function. One is the first case of equation (4) for that we have known very well and another is the second case of equation (4) that is unknown one. Based on structure of solutions and our biological knowledge, the model was classified into eight cases (three categories) by basic properties and discussed requirements for realistic growth curves. Among them only four cases (case 1, 2, 5, and 8) are suitable in modeling tree or stand and biological growth. For equations (10), (11), (12), and (13), the model properties and biological interpretations for parameters are discussed in detail.

All of the theoretical analysis coincided with actual fitting results for the generalized Chapman-Richards function. The fitted curves to *DBH* growth data of cryptomeria plantations with 4 levels of initial spacing, and to the *DBH* and height growth data of the natural Korean pine represent all of four typical growth patterns in trees and stands. They belong to four cases of the generalized Chapman-Richards model. 1) First stand with spacing of 3.1 m×3.1 m (1 024 stems /hm²) belongs to case 1 and the growth curve is sigmoid. 2) For treatments 2 and 3 (1 934 and 3 520 stems /hm²), the growth curves belong to case 2 and it is reversed J-form with asymptotic line and non-inflection point. 3) The stand with spacing of 1.5m×1.5m (4 552 stems /hm²) belongs to case 5 and the growth curve is beyond the asymptotic level with inflection point. 4) Finally, the *DBH* and height growth of the Korean pine tree were described by case 8 and the pattern is more typical sigmoid with asymptotic value and delayed inflection point than case 1. Therefore, the generalized Chapman-Richards model is capable of describing a wide range of typical shapes of growth patterns: asymptotic or nonasymptotic, and inflected or non-inflected trends of growth.

By comparing assumptions, solutions and properties between the generalized Chapman-Richards function and the Schnute model, it is concluded that two models are identical. The differential equations for the two models can be arrived from one to other. As a result, integral solution sets and the expressions for the two models are identical. In fact, the Schnute model is one of the solutions to equation (2) if the latter is used to model a growth process with two known stages. In addition, each case of eight growth curves represents the same region shown in Schnute (1981) and has the same properties for the two models. The crossover point, the asymptotic value and the coordinate of the inflection point, which determine the curve shape, are obviously identical due to the relationships of the parameters between two models. In this study, these conclusions

were verified and confirmed by fitting the models to actual data. Two models fitted the same data set equally well and had the same accuracy in tree and stand growth. Being compared with the Schnute model, the generalized Chapman-Richards function is simpler and more convenient in both formulation and model property. Also, the generalized Chapman-Richards function is very versatile.

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